

4. Advanced Analytical Methods.

4.1. Green's Function Method

4.1.1. Basics of Green's Function Method

We apply Green's function method to solve Nonhomogeneous transient heat conduction problems. It is particularly useful when heat generation (g) and boundary conditions are time dependent.

For a general nonhomogeneous heat conduction problem:

$\nabla^2 T(\vec{r}, t) + \frac{1}{k} g(\vec{r}, t) = \frac{1}{\alpha} \frac{\partial T(\vec{r}, t)}{\partial t}$ B.C. $\left. K \frac{\partial T}{\partial n_i} \right _{S_i} + h_i T \Big _{S_i} = h_i T_{\infty i} \equiv f_i(\vec{r}, t)$ I.C. $T(\vec{r}, t) _{t=0} = F(\vec{r})$	\leftarrow Nonhomo. \leftarrow Nonhomo.
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The Green's function is defined as the solution of the corresponding auxiliary problem ($G(\vec{r}, t | \vec{r}', \tau)$):

$$\nabla^2 G(\vec{r}, t | \vec{r}', \tau) + \frac{1}{k} \delta(\vec{r} - \vec{r}') \delta(t - \tau) = \frac{1}{\alpha} \frac{\partial G(\vec{r}, t | \vec{r}', \tau)}{\partial t} \quad (t > \tau)$$

With homogeneous boundary conditions:

$$K \frac{\partial G}{\partial n_i} \Big|_{S_i} + h_i G \Big|_{S_i} = 0 \quad \leftarrow \text{homogeneous!}$$

and: $G = 0$ for $t < \tau$.

* The physical significance of Green's function (3D):

$G(\vec{r}, t | \vec{r}', \tau)$ represents the temperature at location \vec{r} , at time t , due to an instantaneous point source of unit strength, located at $\vec{r} = \vec{r}'$, releasing its heat at time $t = \tau$.

The delta function:

$\delta(\vec{r} - \vec{r}')$: a point heat source at $\vec{r} = \vec{r}'$.

$\delta(t - \tau)$: an instantaneous source releasing heat at $t = \tau$.

(Note: point source for 3D, line source for 2D, plane source for 1D)

* The practical significance of Green's function:

The solution of the original nonhomogeneous heat conduction problem can be represented in terms of Green's function. Once the Green's function is known, the temperature distribution is readily determined.

Assuming $G(\vec{r}, t | \vec{r}', \tau)$ is known:

$$\boxed{T(\vec{r}, t) = \int G(\vec{r}, t | \vec{r}', \tau) \Big|_{\tau=0} F(\vec{r}') d\vec{r}' + \frac{\alpha}{K} \int_{\tau=0}^t d\tau \int G(\vec{r}, t | \vec{r}', \tau) g(\vec{r}', \tau) d\vec{r}' + \left[\alpha \int_{\tau=0}^t d\tau \sum_i \left(-\frac{\partial G}{\partial n_i} \right) \Big|_{\vec{r}'=\vec{r}_i} \cdot f_i(\vec{r}', \tau) dS'_i \right] \left. \right|_{\vec{r}'=\vec{r}} \leftarrow T|_{S_i} = f_i(\vec{r}, t) + \left[\alpha \int_{\tau=0}^t d\tau \sum_i \left(G \Big|_{\vec{r}'=\vec{r}_i} \cdot \frac{1}{K} f_i(\vec{r}', \tau) dS'_i \right) \right] \left. \right|_{S_i} \leftarrow K \frac{\partial T}{\partial n_i} \Big|_{S_i} = f_i(\vec{r}, t) + \left[\alpha \int_{\tau=0}^t d\tau \sum_i \left(G \Big|_{\vec{r}'=\vec{r}_i} \cdot K f_i(\vec{r}', \tau) dS'_i \right) \right] \left. \right|_{S_i} \leftarrow K \frac{\partial T}{\partial n_i} \Big|_{S_i} + h_i T \Big|_{S_i} = f_i(\vec{r}, t)}$$

The 1st term represents the contribution of the initial condition $F(\vec{r}, t)$,
 the 2nd term represents the contribution of energy generation $g(\vec{r}, t)$,
 the 3rd term represents the contribution of nonhomogeneous B.C. $f_i(\vec{r}, t)$.

4.1.2. Determination of Green's Function.

In order to determine the Green's function, consider the related homogeneous transient heat conduction problem without heat generation:

$\nabla^2 T(\vec{r}, t) = \frac{1}{\alpha} \frac{\partial T(\vec{r}, t)}{\partial t}$	← homogeneous
B.C. $K \frac{\partial T}{\partial n_i} \Big _{S_i} + h_i T \Big _{S_i} = 0$	← homogeneous
I.C. $T(\vec{r}, t) \Big _{t=0} = F(\vec{r})$	remains the same

- * The solution of this homogeneous problem can be derived from separation of variable method! which has the general form:

$$T(\vec{r}, t) = \int K(\vec{r}, \vec{r}', t) \cdot F(\vec{r}') d\vec{r}'$$

- * The solution of this homogeneous problem can also be determined by Green's function method! which can be written as:

$$T(\vec{r}, t) = \int G(\vec{r}, t | \vec{r}', t) \Big|_{t=0} \cdot F(\vec{r}') d\vec{r}'$$

(because $g=0$ and $f_i=0$ for homogeneous problem)

A comparison of the two solution gives:

$$G(\vec{r}, t | \vec{r}', \tau) \Big|_{\tau=0} = K(\vec{r}, \vec{r}', t)$$

Therefore, the solution for the homogeneous transient conduction problem (obtained by separation of variable method) can be rearranged in the form to obtain $G(\vec{r}, t | \vec{r}', 0)$. That is, in order to obtain $G(\vec{r}, t | \vec{r}', 0)$, we can solve the appropriate homogeneous conduction problem.

It can be shown that Green's function $G(\vec{r}, t | \vec{r}', \tau)$ for transient heat conduction can be obtained from $G(\vec{r}, t | \vec{r}', 0)$ by simply replacing "t" by "t-τ".

* Example:

Determine the Green's function for the following nonhomogeneous heat conduction problem of a solid cylinder:

$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{k} g(\vec{r}, t) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$	← Nonhomogeneous t-dependent
B.C. $\left. T \right _{r=0} = \text{finite}$	
$\left. T \right _{r=b} = f(t)$	← Nonhomogeneous t-dependent
I.C. $\left. T \right _{t=0} = F(r)$	

To determine the Green's function, we consider the related homogeneous problem.

$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Psi}{\partial r} \right) = \frac{1}{\alpha} \frac{\partial^2 \Psi}{\partial t^2}$	← homogeneous
B.C. $\left. \Psi \right _{r=0} = \text{finite}$	
$\left. \Psi \right _{r=b} = 0$	← homogeneous
I.C. $\left. \Psi \right _{t=0} = F(r)$	

The solution of this homogeneous problem can be derived by the method of separation of variables.

$$\Psi(\vec{r}, t) = \int_0^b \left[\frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \lambda_m^2 t} \frac{J_0(\lambda_m r) J_0(\lambda_m r')}{J_1^2(\lambda_m b)} \right] F(r') r' dr'$$

$G(r, t | r', 0)$

So:

$$G(r, t | r', 0) \Big|_{t=0} = \frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \lambda_m^2 t} \frac{J_0(\lambda_m r) J_0(\lambda_m r')}{J_1^2(\lambda_m b)}$$

and by replacing "t" by "t-τ":

$$G(r, t | r', \tau) = \frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \lambda_m^2 (t-\tau)} \frac{J_0(\lambda_m r) J_0(\lambda_m r')}{J_1^2(\lambda_m b)}$$